On rotational state of a Bose-Einstein condensate

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Abstract. A quantum liquid of an almost ideal Bose gas brought into rotation is investigated from a physical and mechanical point of view on the basis of the Gross-Pitaevskii (GP) equation by applying a quantum-mechanical scenario, *i.e.* the London's scenario. This scenario allows a superfluid to have rotational states. Considering that the equation governs an interacting Bose gas, it is proposed that the GP equation admits *rotational* flows of a superfluid. This is carried out without incurring essential change of the equation. By this reformulation, a superfluid placed in a rotating vessel is able to have a solid body rotation with the same angular velocity as its container and also to have a meniscus approximated by a parabolic profile. The solid body rotation is accompanied by density increase proportional to the square of its angular velocity. These are consistent with experimental observations. In addition, this formulation allows a vortex of quantized circulation with coaxial rotational core whose density does not necessarily vanish at its center.

Keywords: Superfluid; Gross-Pitaevskii equation; Vorticity; Vortex; Solid-body rotation

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1. Introduction

In view of recent renewing studies of interacting Bose gases in rotation at extremely low temperatures, we consider a possible reformulation of a quantum condensate brought into rotation on the basis of the Gross-Pitaevskii equation without incurring essential change.

Some early experimental studies of rotating helium II (Osborne 1950, Andronikashvili and Kaverkin 1955) showed that the quantum liquid appeared to be rotating uniformly as a whole in a rotating container. The idea of a quantum vortex in rotating superfluid was proposed by Onsager (1949) and Feynman (1955). According to them, the vortex induces a potential flow around it with its circulation quantized and with the density vanishing at its center. At sufficiently fast rotation, the quantum vortices form a lattice that imitates on average the solid body rotation (Andronikashvili *et al* 1961, Andronikashvili and Mamaladze 1966). This is understood as follows. The rotating superfluid is threaded by a lattice of quantized *potential* vortices in a manner which rotates as a whole (Donnelly 1991). In §2 of this paper, we reconsider whether the vortices are characterized by potential flows only and propose possibility of having a rotational core.

Although their approach is different from the present study, the same subject of possibility for a quantum liquid to be in a rotational state was studied recently by Greenberg and Zelevinsky (2012). Our approach however is motivated by a recent new representation of rotational flows of an ordinary fluid (Kambe 2013), which satisfies Euler's equation of motion of a compressible ideal fluid. Note that the system of equations derived from the Gross and Pitaevskii (Gross 1961, Pitaevskii 1961) have analogous form to that of rotational flows of an ordinary inviscid fluid.

In the present study we consider a quantum liquid, *i.e.* a degenerate state of an interacting Bose gas of mass m, which is governed by the Gross-Pitaevskii equation. On the basis of the Lagrangian formulation of particle mechanics in rotating frame and also corresponding formulation of quantum mechanics, we find a certain type of rotational state that does not necessarily result in formation of potential vortices (but a solution of the potential vortex is not excluded). Section 3 investigates its formulation. Some examples of solution resulting from the formulation are presented for solid-body rotation of a quantum liquid in §4 and for quantum vortices with coaxial rotational core in §5.

1.1. Superfluids

Firstly we review past studies on the superfluids brought into rotation, and pose a question whether the theory of potential vortices only can explain all. We consider possibility if rotational state is allowed for *a degenerate almost ideal Bose gas* at absolute zero within the current framework of the Gross-Pitaevskii system. (For a degenerate almost ideal Bose gas, see Lifshitz and Pitaevskii (1980) §25.)

(a) Regarding the solid-body rotation of superfluids, there exist some experimental evidences. Tkachenko (1966) showed that the superfluid behaved like a fluid rotating with the velocity of a solid body at a macroscopic level. There is a critical angular velocity Ω_c of a rotating vessel for vortex formation. General understanding of a superfluid in a fast rotating vessel is that the rotating liquid forms a lattice of quantum vortices.

By their experimental study, Andronikashvili and Tsakadze (1965a, b) showed that the helium II (*i.e.* the liquid helium below the λ-point temperature) brought into motion in a rotating vessel *increases* its density appreciably. This fact urges more careful study of rotating quantum condensates because the structure of vortex lattice with density defect at the vortex centers implies reduction of the density of helium II. There must be an unknown effect that compensates the density reduction at the lattice points if the theory of potential vortices is the only valid one. In regard to the helium I of normal fluid, appreciable increase of the density is not observed by the increase of angular velocity. Section 4 gives a clue to this problem.

(*b*) Experimental investigations of rotating helium II in early times (Osborne 1950, Andronikashvili and Kaverkin 1955, Donnelly *et al* 1956) showed that the quantum liquid He II in a rotating vessel formed a meniscus as if it is a normal fluid undergoing solid-body rotation. In fact, the meniscus, *i.e.* the shape of free surface, of rotating helium II was given by the same parabolic height as that of the normal fluid with respect to the distance from the rotation axis. This implies that the superfluid component is participating in the rotation. This was also observed in the experiments of rotating He II by Hall and Vinen (1955). Measurement of the angular momentum of rotating helium II showed also that the liquid motion corresponds to the solid body rotation (Reppy *et al* 1960). Kiknadze *et al* (1965) showed that two-dimensional vortex lattices in rotating helium II are capable of rotation together with its rotating vessel.

(*c*) Note that the study of Kiknadze *et al* (1965) took analogy with the theory of type II superconductors of Abrikosov (1957). A type-II superconductor is characterized by the formation of magnetic vortices in an applied magnetic field. This occurs above a certain critical strength H_c of applied magnetic field. The theory of type-II superconductor in magnetic field was developed by Abrikosov, who applied the ideas of Onsager-Feynman's quantum vortices of superfluids to the lines of magnetic flux passing through the material. The region of lines of magnetic flux is surrounded by a circulating supercurrent of electrons. In analogy with the superfluid, the swirling supercurrent creates the so-called *Abrikosov vortex*. He found that the vortices arrange themselves into a regular array. Density distribution of superconducting electron coincides exactly with that of a superfluid in corresponding regular array of vortices. There is a close analogy between the Abrikosov vortex of type-II superconductor and the Onsager-Feynman vortex of superfluid.

In the case of type-II superconductor, a flux of magnetic lines of force passes through the superconducting material. It would be interesting to consider whether this implies that a flux of vortex lines passes through the superfluid and forms a vortex of quantized circulation with a rotational core. In this regard, Andronikashvili and Kavelkin (1955), cited by Andronikashvili and Mamaladze (1966), describes their experiment of a helium II in a container which was brought into rotation from the state at rest. Dragging of the fluid into rotation is observed to be different between He I (ordinary fluid) and He II (superfluid). In the case of He II, *i.e.* the quantum condensate, only its peripheral layers were dragged into rotation at first, while the central part of the meniscus remained flat.

Gradually the radius of the flat part of the meniscus became smaller and at last a parabolic free surface was formed. It was indistinguishable from the parabolic meniscus of a classical viscous liquids at moderate velocities of rotation. This observation implies that vertical vortex lines gradually penetrate into the liquid helium from the peripheral layers inward. This reminds us of the penetration of magnetic lines in the type-II superconductor. They concludes at the end that *the superfluid component maintains motion for which* curl $\boldsymbol{v}_s$ *differs from zero* (where $\boldsymbol{v}_s$ denotes the velocity of superfluid component).

Section 5 presents an example of quantum vortices with coaxial rotational core, whose density does not vanish at its center. Jackson *et al* (2009) and also Allen *et al* (2013) studied *finite-temperature effect* on vortex dynamics and found non-vanishing density at the vortex core which is a thermal cloud owing to the finite temperature. However, in the present case, the rotational core is obtained by solving the Gross-Pitaevskii equation at the absolute zero temperature.

1.2. London's scenario

Thus in this paper, we consider how a quantum condensate is endowed mathematically with rotational state. In early times of quantum mechanics, London (1927) proposed a scenario to improve Schrödinger's quantum mechanics and succeeded in introducing electromagnetism into the Schrödinger system [see O'Raifeartaigh (1997)]. We consider analogous scenario to endow a superfluid with rotational property. Although it is usually considered that the Gross and Pitaevskii equation (GP equation in short) implies potential flows of superfluid, we investigate a possibility that a quantum liquid governed by the GP equation may support *rotational* motion. By the London's scenario applied to the GP equation, it is shown in §2.3 that the same equation can describe rotational motions of superfluid. This is not surprising because the non-linearity of the Gross-Pitaevskii equation has its origin in the interaction between two particles in the condensate.

2. Gross-Pitaevskii equation

We consider a quantum condensate of almost ideal Bose gas of atomic mass m. There exist interactions between atoms, represented by a short-range interatomic potential $U_0\,\delta(\boldsymbol{x})$, which is repulsive if U_0 is positive. The coefficient U_0 is given by $U_0 = 4\pi\hbar^2 a/m$ with a the s-wave scattering length and $\hbar = h/2\pi$ with h the Planck constant. In such a slightly non-ideal Bose gas at absolute zero temperature, almost all the particles, but not all, are in the condensate. A wave function operator is written as $\hat{\Psi}(\boldsymbol{x},t) = \Psi(\boldsymbol{x},t) + \Psi'(\boldsymbol{x},t)$ where $\Psi = \langle\hat{\Psi}\rangle$ is the *order parameter* (the symbol $\langle\,\cdot\,\rangle$ denoting ensemble average). The condensate wave function Ψ describes the mode that has macroscopic occupation with $|\Psi|^2 = n(\boldsymbol{x})$ giving the condensate particle number density at $\boldsymbol{x}$.

2.1. Equation of a condensate of almost ideal Bose gas

If only pair interactions are taken into account, the equation governing Ψ is given by the Gross-Pitaevskii equation. In fact, in a trap with confining external potential V_e, the

Gross-Pitaevskii energy functional for the condensate (Fetter 2009) is defined by

$$\mathcal{E}_{GP} = \int \left(\frac{\hbar^2}{2m} |\nabla\Psi|^2 + V_e |\Psi|^2 + \tfrac{1}{2} U_0 |\Psi|^4 \right) \mathrm{d}^3\boldsymbol{x}. \tag{1}$$

The total particle number is defined by $N = \int |\Psi|^2 \mathrm{d}^3\boldsymbol{x}$. Minimizing $\mathcal{E}_{GP} - \mu N$ under the constraint condition $N = const$, we obtain the following *Gross-Pitaevskii* equation:

$$-(\hbar^2/2m)\,\nabla^2\Psi + \Big(U_0 |\Psi|^2 + V_e - \mu \Big) \Psi = 0, \tag{2}$$

(Ginzburg and Pitaevskii 1958, Gross 1961, Pitaevskii 1961), where μ is a chemical potential. In the uniform stationary state of $|\Psi| = \sqrt{n_0} = const$ in the constant negative potential $V_e = -\eta$ (where $\eta > 0$), in which $\nabla\Psi = 0$, it is immediately seen from (2) that we have $U_0 n_0 = \mu + \eta$. The Gross-Pitaevskii (GP) equation assumes: (i) there exist a large macroscopic number of Bose atoms in the ground state (*i.e.* in the condensate) at absolute zero temperature, (ii) $\langle \Psi'(\boldsymbol{x}, t) \rangle$ is negligible in comparison with $\Psi = \langle \hat{\Psi} \rangle$, and ($iii$) the inter-particle spacing ($\sim n_0^{-1/3}$) is much larger than a. The conditions (i) and (ii) are referred to as those for a slightly non-ideal or *almost ideal* Bose gas.

2.2. Time-dependent Gross-Pitaevskii equation

The equation (2) can be generalized to the following time-dependent GP equation:

$$i\hbar\,\partial_t\Psi = -(\hbar^2/2m)\,\nabla^2\Psi + \Big(U_0 |\Psi|^2 + V_e \Big) \Psi, \tag{3}$$

where $\partial_t \equiv \partial/\partial t$, and Ψ now depends on t as well as on $\boldsymbol{x}$.§

The non-idealness of the Bose gas causes the presence of particles with non-zero momentum even at absolute zero because of the interaction potential $U_0\,\delta(\boldsymbol{x})$ (see *e.g.* Lifshitz and Pitaevskii (1980) §25).‖ This GP picture remarkably fits to the system of dilute ultracold trapped alkali-metal gases, developed since 1995.

Bogoliubov (1947) introduced an essential mechanism of such a system of a dilute Bose gas of repulsive interactions (see Lifshitz and Pitaevskii (1980) Chap.III). It is particularly remarkable that the existence of superfluidity in a uniform dilute Bose gas arises indeed from the repulsive interactions, since the Landau critical velocity is given by $v_c = \sqrt{U_0 n_0/m}$, which vanishes for $U_0 = 0$ (Fetter 2009).

Expressing Ψ with a complex polar form $\Psi = |\Psi| \exp(i\varphi)$, substituting it in (3), and dividing the resulting expression into imaginary and real parts, we have

$$2\partial_t|\Psi| + 2\frac{\hbar}{m}\nabla|\Psi|\cdot\nabla\varphi + \frac{\hbar}{m}|\Psi|\nabla^2\varphi = 0\,, \tag{4}$$

$$\hbar\,\partial_t\varphi + \frac{\hbar^2}{2m}(\nabla\varphi)^2 - \frac{\hbar^2}{2m}\frac{\nabla^2|\Psi|}{|\Psi|} + V_e + U_0|\Psi|^2 = 0\,. \tag{5}$$

The mass probability density ρ is defined by $\rho = m|\Psi|^2$. Introducing a function Φ by $\varphi = (m/\hbar)\Phi$, a current vector of velocity $\boldsymbol{u}$ can be defined by

$$\boldsymbol{u} = \nabla\Phi, \qquad \Phi \equiv (\hbar/m)\,\varphi. \tag{6}$$

§ Comparison of (2) and (3) implies that a stationary solution has the time factor $\exp[-i\mu t/\hbar]$.

‖ By collision of two particles of momenta $\boldsymbol{p}_1 = 0$ and $\boldsymbol{p}_2 = 0$ in the condensate of non-ideal Bose gas, there exists non-zero probability of transition to new states of $\boldsymbol{p}_1' = \boldsymbol{p} \neq 0$ and $\boldsymbol{p}_2' = -\boldsymbol{p}$, respectively.

Then the first equation (multiplied by $m|\Psi|$) can be transformed to

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{7}$$

i.e. the continuity equation. The second (divided by m) can be rewritten as

$$\partial_t \Phi + \tfrac{1}{2}\,\boldsymbol{u}^2 + h_{\rm nl} + h_{\rm qm} + V_e/m = 0\,, \tag{8}$$

$$h_{\rm nl} = (U_0/m^2)\,\rho, \quad h_{\rm qm} = -(\hbar^2/2m^2)(|\Psi|^{-1}\,\nabla^2|\Psi|). \tag{9}$$

2.3. Non-integrable factor to represent rotational flows

In quantum mechanics, the *electromagnetism* is introduced into the Schrödinger system. This was first proposed by London (1927) by applying an ingenious mathematics to reformulate the Schrödinger equation. In this subsection we attempt to introduce *rotational* property by applying an analogous mathematical scenario to the GP system of superfluid. An alternative formulation is given in the next section §3.

In quantum mechanics, a momentum $\boldsymbol{p}$ is represented by the operator $-i\hbar\nabla$. However, the motion of a charged particle (of electric charge e) in an electromagnetic field (represented by a vector potential $\boldsymbol{A}(\boldsymbol{x},t)$), the momentum should be replaced by

$$\boldsymbol{p} - \frac{e}{c}\boldsymbol{A} = -i\hbar\nabla - \frac{e}{c}\boldsymbol{A} = -i\hbar\left(\nabla - i\frac{e}{\hbar c}\boldsymbol{A}\right), \tag{10}$$

(Landau and Lifshitz 1977 §111). This is equivalent to the following replacement:

$$\nabla \to \mathcal{D}_A \equiv \nabla - i\frac{e}{\hbar c}\boldsymbol{A} \qquad \text{(a covariant derivative).} \tag{11}$$

Namely, the space derivative ∇ should be replaced by the covariant derivative $\nabla - i(e/\hbar c)\boldsymbol{A}$, together with the time derivative ∂_t replaced by $\partial_t + i(e/\hbar)\phi$ (where ϕ is an electric scalar potential). For the Schrödinger system represented by the equation such as (3), the above replacement is equivalent to the replacement of Ψ:

$$\Psi(x) \;\to\; \exp\left[i\frac{e}{\hbar c}\,\Phi^*\right]\,\Psi(x), \qquad {\rm d}\Phi^* = -A_\sigma\,{\rm d}x^\sigma, \tag{12}$$

where four vectors are introduced by $A_\sigma = (-\phi, \boldsymbol{A})$ and $x^\sigma = (ct, \boldsymbol{x})$ with $\boldsymbol{A}$ and $\boldsymbol{x}$ three vectors. In fact, the factor $(e/\hbar c)\,\Phi^*$ is called the London's phase factor introducing *electromagnetism* into the Schrödinger system (O'Raifeartaigh 1997).

Using structural analogy of the present system with the Schrödinger system, we follow the formulation of quantum mechanics. It is observed that the current field would not be singly connected when there exists a potential vortex, because its axis is characterized as a line singularity. In order to accommodate vortices, it is proposed that the wave function $\Psi = |\Psi|\,\exp[i(m/\hbar)\Phi]$ of (3) is extended to Ψ^* with an extended phase factor $(m/\hbar)\Phi^*$ in stead of $(m/\hbar)\Phi$, where

$$\Psi^* \;=\; \exp\left[i\frac{m}{\hbar}\,\Phi^*\right]\,\Psi(x) = |\Psi|(t,\boldsymbol{x})\;\exp\left[i\frac{m}{\hbar}\,\Theta\right], \tag{13}$$

$${\rm d}\Theta = {\rm d}\Phi + {\rm d}\Phi^*, \qquad {\rm d}\Phi^* = w_1\,{\rm d}x^1 + w_2\,{\rm d}x^2 + w_3\,{\rm d}x^3, \tag{14}$$

and ${\rm d}\Phi = \partial_1\Phi\,{\rm d}x^1 + \partial_2\Phi\,{\rm d}x^2 + \partial_3\Phi\,{\rm d}x^3$. The expression of ${\rm d}\Theta$ implies that the current density $\boldsymbol{v}_c \equiv (i\hbar/2m)(\Psi^*\nabla\bar{\Psi}^* - \bar{\Psi}^*\nabla\Psi^*)$ is given by $|\Psi|^2\nabla\Theta = |\Psi|^2(\nabla\Phi + \boldsymbol{w})$, and the generalized momentum $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = m\boldsymbol{v}_c = \rho\,\boldsymbol{v}, \qquad \boldsymbol{v} \equiv \nabla\Phi + \boldsymbol{w}, \quad \boldsymbol{w} = (w_1, w_2, w_3). \tag{15}$$

It is found that the potential velocity $\boldsymbol{u} = \nabla\Phi$ of (6) is now replaced by a rotational velocity $\boldsymbol{v}$. In place of (6)∼(8), we have an extended system of equations:

$$\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}, \quad \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0, \quad \partial_t\Phi + \tfrac{1}{2}\,\boldsymbol{v}^2 + h_{\rm nl} + h_{\rm qm} + V_e/m = const\,, \tag{16}$$

where $\boldsymbol{w} = (w_1(\boldsymbol{x}),\, w_2(\boldsymbol{x}),\, w_3(\boldsymbol{x}))$. Here, we define a differential one-form $\mathcal{V}^{(1)}$ by $v_1\,\mathrm{d}x^1 + v_2\,\mathrm{d}x^2 + v_3\,\mathrm{d}x^3$. If $\mathcal{V}^{(1)}$ is an exact differential (by $\boldsymbol{w} = 0$), *i.e.* if $\mathcal{V}^{(1)} = \mathrm{d}\Phi$, then it defines the potential velocity $\boldsymbol{v} = \nabla\Phi$. However, if $\mathcal{V}^{(1)}$ is not an exact differential, namely if

$$\mathrm{d}\mathcal{V}^{(1)} = \sum \Big(\partial_j w_k - \partial_k w_j\Big)\,\mathrm{d}x^j \wedge \mathrm{d}x^k\Big|_{j<k} \neq 0,$$

we have a rotational flow $\boldsymbol{v}$ of vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{w}$, since its i-th component is $\omega_i = \epsilon_{ijk}\partial_j w_k\ (= \partial_j w_k - \partial_k w_j)$.

In this reformulation, the original momentum $-i\hbar\nabla(i\varphi) = m\nabla\Phi$ is replaced by

$$-i\hbar\nabla[(im/\hbar)\Theta] = -i\hbar[\nabla(i\varphi) + i(m/\hbar)\,\boldsymbol{w}] = m\nabla\Phi + m\,\boldsymbol{w}.$$

Equivalently, the ∇ operator of (3) is replaced by the covariant derivative $\mathcal{D}_w$:

$$\nabla \to \mathcal{D}_w \equiv \nabla + i\frac{m}{\hbar}\,\boldsymbol{w} \qquad \text{(a covariant derivative).} \tag{17}$$

This is analogous to (11) of the electromagnetic case, and $m\boldsymbol{w}$ corresponds to the electromagnetic momentum $-(e/c)\boldsymbol{A}$. Note that the vector $\boldsymbol{w}$ denotes a current velocity of rotational motion. Thus, a rotational component is introduced into the GP system. This implies that the GP equation can support rotational motion in general.

3. Rotational state of an almost ideal Bose condensate

We consider extension of the GP equation to that for a rotating frame with a different approach. Beginning with the particle mechanics of §3.1, we extend it to its quantum mechanical formulation for a rotating frame and in §3.2 apply it to description of a rotational state of an interacting Bose gas. Here, an important remark is to be made from a physical point of view on possible macroscopic motions. Namely, a closed system in thermodynamic equilibrium can take only two states: uniform translation or *uniform rotation* (Landau and Lifshitz (1980) §10). Therefore, we consider possibility of solid body rotation of a macroscopic condensate, in addition to a static state or uniform translation.

3.1. Mechanics of a point mass in a rotating frame

Suppose that a particle of mass m is moving with a velocity $\boldsymbol{v}_0$ in a potential field V in a fixed inertial frame of reference K_0. In the mechanics, the Lagrangian of the particle motion is given by $\mathcal{L}_0 = \frac{1}{2}\,m\boldsymbol{v}_0^2 - V$.¶ The momentum and energy of the particle are

$$\boldsymbol{p}_0 = \partial\mathcal{L}_0/\partial\boldsymbol{v}_0 = m\boldsymbol{v}_0, \qquad \mathcal{E}_0 = \boldsymbol{p}_0\cdot\boldsymbol{v}_0 - \mathcal{L}_0 = \tfrac{1}{2}\,m\boldsymbol{v}_0^2 + V,$$

¶ By the particle trajectory $q^i(t)$, the velocity and momentum are given by $v_0^i = \dot{q}^i$ and $p_0^i = \partial\mathcal{L}_0/\partial\dot{q}^i$.

in the frame K_0. Concerning the same motion, let us take another frame of reference K which is rotating relative to K_0 with an angular velocity $\boldsymbol{\Omega}$ around the common origin. The particle velocity is now $\boldsymbol{v}_K$ relative to K, and $\boldsymbol{v}_0$ is composed of two components:

$$\boldsymbol{v}_0 = \boldsymbol{v}_K + \boldsymbol{\Omega} \times \boldsymbol{x} = \boldsymbol{v}_K + \boldsymbol{w}_\Omega, \qquad \boldsymbol{w}_\Omega \equiv \boldsymbol{\Omega} \times \boldsymbol{x},$$

where $\boldsymbol{w}_\Omega$ is the velocity of the frame rotation at the position $\boldsymbol{x}$. The Lagrangian in K is

$$\mathcal{L}_K = \tfrac{1}{2}\, m\boldsymbol{v}_K^2 + m\boldsymbol{v}_K \cdot \boldsymbol{w}_\Omega + \tfrac{1}{2}\, m\boldsymbol{w}_\Omega^2 - V. \tag{18}$$

Generalized momentum $\boldsymbol{p}_K$ and energy $\mathcal{E}_K$ in the frame K are

$$\boldsymbol{p}_K \equiv \frac{\partial \mathcal{L}_K}{\partial \boldsymbol{v}_K} = m\boldsymbol{v}_K + m\boldsymbol{w}_\Omega = m\boldsymbol{v}_0, \quad \mathcal{E}_K \equiv \boldsymbol{p}_K \cdot \boldsymbol{v}_K - \mathcal{L}_K = \tfrac{1}{2}\, m\boldsymbol{v}_K^2 - \tfrac{1}{2}\, m\boldsymbol{w}_\Omega^2 + V. \tag{19}$$

Thus, the momentum $\boldsymbol{p}_K$ and angular momentum $\boldsymbol{M}_K = \boldsymbol{x} \times \boldsymbol{p}_K$ are the same as $\boldsymbol{p}_0$ and $\boldsymbol{M}_0 = \boldsymbol{x} \times \boldsymbol{p}_0$, while $\mathcal{E}_K$ is different from $\mathcal{E}_0$. Substituting $\boldsymbol{v}_K = \boldsymbol{v}_0 - \boldsymbol{w}_\Omega$, we obtain

$$\mathcal{E}_K = \tfrac{1}{2}\, m\boldsymbol{v}_0^2 + V - m\boldsymbol{v}_0 \cdot (\boldsymbol{\Omega} \times \boldsymbol{x}) = \mathcal{E}_0 - \boldsymbol{\Omega} \cdot \boldsymbol{M}_0.$$

Thus the energy is transformed according to $\mathcal{E}_K = \mathcal{E}_0 - \boldsymbol{\Omega} \cdot \boldsymbol{M}_0$.

3.2. Equation of order parameter in a rotating frame

We try to extend the energy functional (1) of the superfluid to that in a rotating frame. A superfluid in the rotating frame is described by an order parameter ψ determined by minimizing an energy functional corresponding to the energy $\mathcal{E}_0 - \boldsymbol{\Omega} \cdot \boldsymbol{M}_0$ (Lifshitz and Pitaevskii (1980) §29). The mechanical energy $\mathcal{E}_0 - \boldsymbol{\Omega} \cdot \boldsymbol{M}_0$ can be rewritten as

$$\tfrac{1}{2}\, m\boldsymbol{v}_0^2 + V - \boldsymbol{\Omega} \cdot (\boldsymbol{x} \times m\boldsymbol{v}_0) = \frac{1}{2m}(m\boldsymbol{v}_0 - m\boldsymbol{w}_\Omega)^2 - \tfrac{1}{2}\, m\boldsymbol{w}_\Omega^2 + V. \tag{20}$$

It is noted that the term in the parenthesis on the right hand side is the linear momentum $m\boldsymbol{v}_K$ in the rotating frame K:

$$m\boldsymbol{v}_0 - m\boldsymbol{w}_\Omega = m\boldsymbol{v}_K. \tag{21}$$

In quantum mechanics, the linear momentum $\boldsymbol{p} = m\boldsymbol{v}_0$ and angular momentum $\boldsymbol{M} = \boldsymbol{x} \times m\boldsymbol{v}_0$ are expressed by $\boldsymbol{p} = -i\hbar\nabla$ and $\boldsymbol{M} = -i\hbar\boldsymbol{x} \times \nabla$, respectively. Therefore, the above relation (21) may be written quantum-mechanically as

$$-i\hbar\nabla_0 - m\boldsymbol{w}_\Omega = -i\hbar\nabla_K, \quad \text{equivalently} \quad \nabla_K = \nabla_0 - i\frac{m}{\hbar}\boldsymbol{w}_\Omega. \tag{22}$$

The symbols ∇_0 and ∇_K are the nabla's with respect to the frames K_0 and K. Let us extend the energy $\mathcal{E}_{GP}$ of (1) to that of a rotating frame. Corresponding to the right-hand side of (20), a quantum-mechanical energy $\mathcal{E}_{GPr}$ in a rotating frame is given by

$$\mathcal{E}_{GPr} = \int \mathrm{d}^3\boldsymbol{x} \Big[\, \frac{1}{2m}\,(-1)^2 \Big|(i\hbar\nabla_0 + m\boldsymbol{w}_\Omega)\,\psi\Big|^2 + \Big(-\tfrac{1}{2}\, m\boldsymbol{w}_\Omega^2 + V_e\Big)\,|\psi|^2 + \tfrac{1}{2}\, U_0\,|\psi|^4 \,\Big], \tag{23}$$

where V_e is used in place of V, and the self-interaction term $\frac{1}{2}\, U_0\,|\psi|^4$ is added. This form is used in the studies of bosons in rapid rotation by Fischer and Baym (2003), and by Correggi *et al* (2007). On the other hand, corresponding to the left-hand side of (20), we have an alternative expression $\mathcal{E}'_{GPr}$ for the quantum-mechanical energy:

$$\mathcal{E}'_{GPr} = \int \mathrm{d}^3\boldsymbol{x} \Big[\, \frac{1}{2m}\,|-i\hbar\,\nabla_0\,\psi|^2 + V_e\,|\psi|^2 + \tfrac{1}{2}\, U_0\,|\psi|^4 \,\Big] - \boldsymbol{\Omega} \cdot \int \tilde{\psi}\,\boldsymbol{M}\,\psi\; \mathrm{d}^3\boldsymbol{x}, \tag{24}$$

where $\tilde{\psi}$ is the complex conjugate of ψ.

In the non-rotating frame, the expression (23) reduces to (1) since $\boldsymbol{w}_\Omega = 0$. Variation of $\mathcal{E}_{GPr} - \mu N$ under the subsidiary condition of fixed $N = \int |\psi|^2 \mathrm{d}^3\boldsymbol{x}$ yields

$$\frac{1}{2m}\,(i\hbar\,\nabla_0 + m\boldsymbol{w}_\Omega)^2\psi + \Big(-\tfrac{1}{2}\,m\boldsymbol{w}_\Omega^2 + V_e\Big)\,\psi + U_0\,|\psi|^2\,\psi = \mu\psi. \tag{25}$$

This is the GP equation in a rotating frame with the angular velocity $\boldsymbol{\Omega}$. An equivalent equation was given by Yngvason (2008). This reduces to (2) in the rest frame when $\boldsymbol{w}_\Omega = 0$. In view of (22), the equation (25) can be written as

$$-(\hbar^2/2m)\,\nabla_K^2\psi - \tfrac{1}{2}\,m\boldsymbol{w}_\Omega^2\,\psi + \Big(V_e + U_0\,|\psi|^2\Big)\,\psi = \mu\psi. \tag{26}$$

Since ∇_K^2 is a scalar product of ∇_K operator, we have invariance of the scalar $\nabla_K^2 = \nabla_0^2$ by rotational transformation between two frames K_0 and K, which can be written simply as ∇^2. Thus, we have the equation governing a quantum condensate in the frame rotating with $\boldsymbol{w}_\Omega = \boldsymbol{\Omega}\times\boldsymbol{x}$:

$$-(\hbar^2/2m)\,\nabla^2\psi - \tfrac{1}{2}\,m\boldsymbol{w}_\Omega^2\,\psi + \Big(V_e + U_0\,|\psi|^2\Big)\,\psi = \mu\psi. \tag{27}$$

In §4, this equation will be applied to two problems related to the issues (a) and (b) of §1.1, *i.e.* density increase and parabolic meniscus of a condensate in a rotating vessel.

3.3. Extension to a rotational quantum liquid

Now, we attempt to generalize the formulation to rotational states in general, whose velocity $\boldsymbol{v}$ is composed of a potential component $\nabla\Phi$ and rotational component $\boldsymbol{w}_r$: *i.e.* $\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}_r$. Its vorticity $\boldsymbol{\omega}(\boldsymbol{x}) = \nabla\times\boldsymbol{w}_r(\boldsymbol{x})$ represents local rotation of the fluid with the angular velocity $\boldsymbol{\Omega}^*(\boldsymbol{x}) = \frac{1}{2}\,\boldsymbol{\omega}(\boldsymbol{x})$ at a point $\boldsymbol{x}$. We apply the principle of *local gauge symmetry* to the rotational flow. The idea is as follows.

Consider a collection of local frames K^* rotating with local angular velocity $\boldsymbol{\Omega}^*(\boldsymbol{x})$ at each point $\boldsymbol{x}$ of the flow field. By the principle of local gauge symmetry[+], the expression of the first two terms of (26), defined by Q_2, would be $Q_2 = -(\hbar^2/2m)\,\nabla_{K^*}^2\psi - \frac{1}{2}\,m(\boldsymbol{w}_{\Omega^*})^2\,\psi$ where $\boldsymbol{w}_{\Omega^*}$ is the velocity of local rotation. To transform it back to non-rotating frame, we note that, relative to the frame K^*, the non-rotating frame is rotating with the velocity $-\boldsymbol{w}_{\Omega^*}$. Hence the transformed Q_2 is expressed by $-(\hbar^2/2m)\,(\nabla + i(m/\hbar)\boldsymbol{w}_{\Omega^*})^2\psi$ (by the second of (22) where ∇_0 is written simply as ∇ and $\boldsymbol{w}_\Omega$ is replaced by $-\boldsymbol{w}_{\Omega^*}$). The last expression can be written as $-(\hbar^2/2m)\,(\nabla + i(m/\hbar)\boldsymbol{w}_r)^2\psi$, because the difference $\boldsymbol{w}_r - \boldsymbol{w}_{\Omega^*}$ is irrotational and can be absorbed in ∇. Thus, instead of the operator ∇ of (3), we propose the covariant derivative $\mathcal{D}_w$ of (17) for rotational states under the principle of local gauge symmetry:

$$\nabla \to \mathcal{D}_w = \nabla + i\frac{m}{\hbar}\boldsymbol{w}_r. \tag{28}$$

[+] The equation (26) is valid in the rotating frame K where all the points of K are related to those of K_0 by rotational transformation as a whole under the common angular velocity $\boldsymbol{\Omega}$. This is called the *global* rotational transformation. The equation is said to have *global gauge symmetry*. The gauge principle requires *local gauge symmetry* as well for local rotational transformation under local angular velocity $\boldsymbol{\Omega}^*(\boldsymbol{x})$. The *local* means *point-wise* in this case.

Thus for rotational states, the GP equation (3) is transformed to

$$i\hbar\,\partial_t\Psi = -(\hbar^2/2m)\,(\nabla + i\frac{m}{\hbar}\boldsymbol{w}_r)^2\Psi + \Big(V_e + U_0\,|\Psi|^2\Big)\,\Psi, \tag{29}$$

under an external potential V_e. Instead of (2), its time-independent version is given by

$$-(\hbar^2/2m)\,(\nabla + i\frac{m}{\hbar}\boldsymbol{w}_r)^2\psi + \Big(V_e + U_0\,|\psi|^2 - \mu\Big)\,\psi = 0. \tag{30}$$

In §5, this equation will be applied to a problem related to the issue (c) of §1.1, *i.e.* a quantized vortex with coaxial rotational core.

4. Density change by rotation

By their experiment, Andronikashvili and Tsakadze (1965a, b) showed that the quantum liquid He II brought into motion in a rotating vessel increases its density in proportion to nearly square of angular velocity Ω of the rotation. This is not recognized sufficiently by most texts in general. The equation (27) gives us some insight into this observed property because it is the equation with respect to a rotating frame.

Suppose that the condensate is at rest in a rotating cylinder of radius R without forming vortices. Then the wave function ψ of (27) should be *real* and approximated by a linear behavior $b_1 r$, where r is the radial distance from the rotation axis with b_1 a constant (this is checked just below). Then from the equation (27), we obtain

$$V_e + U_0\,|\psi|^2 \approx \tfrac{1}{2}\,m\Omega^2 r^2 + \mu + (\xi/r)^2\,U_0 n_0, \tag{31}$$

$$\xi \equiv \frac{\hbar}{\sqrt{2mU_0 n_0}}, \tag{32}$$

where ξ is the healing length. The last term of (31) came from $(\hbar^2/2m)\nabla^2\psi/\psi$, where $\nabla^2\psi/\psi$ is replaced with $(b_1 r)^{-1}\nabla^2(b_1 r) = 1/r^2$, and the coefficient $(\hbar^2/2m)$ is replaced with $\xi^2 U_0 n_0$ by using (32). The last term may be neglected if $r \gg \xi$. Then, the right hand side tends to be proportional to r^2 as r becomes much larger than $\sqrt{2\mu/m}\,\Omega^{-1}$. This is consistent with the original assumption $\psi \propto r$ when the external potential V_e is a constant and $(b_1 r)^2 \gg |V_e|/U_0$. The term $|\psi|^2$ on the left hand side is the particle number density n. Hence, $n = |\psi|^2 \propto \Omega^2 r^2$ for large r, which is consistent with the observation of Andronikashvili and Tsakadze, noted above.

Next, we consider that V_e is the gravitational potential $V_e = mgz$. On the free surface, *i.e.* on the meniscus $z = z_m$ of the condensate, we have $|\psi|(z = z_m) = 0$. Then we obtain $z_m \approx (\Omega^2/2g)\,r^2 + const$ from (31). This is consistent with the experiment of Andronikashvili and Kaverkin (1955), observing that the depth of the meniscus did not depart from the parabolic depth of normal liquids.

5. A quantized vortex with rotational core

Next example is a quantized vortex with a coaxial rotational core. The wave function is expressed by $\psi = |\psi(r)|\,e^{i\kappa\varphi}$ in the cylindrical frame (z, r, φ). In the case of a single quantization (*i.e.* $\kappa = 1$), we have $\psi = |\psi(r)|\,e^{i\varphi}$ for a quantized vortex (having a rotational core of a size δ). Namely, far from the vortex core (for $r/\delta \gg 1$), the flow tends to a potential flow asymptotically, expressed by a velocity potential $\Phi = \gamma\varphi$ of a quantized circulation $2\pi\gamma = h/m$ (*i.e.* $\boldsymbol{v} \to \nabla\Phi$). Its rotational core is assumed to have a velocity $\boldsymbol{w}_r$ represented by a Gaussian profile:

$$\boldsymbol{w}_r = (0, 0, w_\varphi), \qquad w_\varphi = -\frac{\gamma}{r}\exp\Big[-\frac{r^2}{\delta^2}\Big], \qquad \gamma = \hbar/m,$$

where δ is a scale of the rotational core. Total velocity and vorticity are

$$\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}_r = (0,\,0,\gamma\,W_*(r)\,), \quad W_*(r) = \frac{1}{r} - \frac{1}{r}\,e^{-\eta^2}, \quad \eta \equiv r/\delta, \tag{33}$$

$$\boldsymbol{\omega} = (\,\omega_c(r),\,0,\,0), \qquad \omega_c(r) = \gamma\,\frac{1}{r}\,\partial_r(r\,W_*) = \frac{2\gamma}{\delta^2}\,e^{-\eta^2}. \tag{34}$$

This represents a quantized vortex with *a rotational core* of strength $2\pi\gamma$. Close to the center, total velocity is expressed as $\gamma W_*(r) = (\gamma/\delta^2)\,[r + (r^3)]$, which tends to 0 as $r \to 0$. Thus, the velocity is regularized at the center, while the velocity of a potential vortex diverges like r^{-1} as $r \to 0$.

5.1. Governing equation

Governing equation of ψ is given by (30). Substituting $\psi = |\psi|\,e^{i\varphi}$, the imaginary part (of the coefficient of $e^{i\varphi}$) describes the continuity equation as before, and the real part is

$$\nabla^2|\psi| - W_*^{\,2}(r)\;|\psi| + \frac{2m\mu_*}{\hbar^2}\,|\psi| - \frac{2mU_0}{\hbar^2}\,|\psi|^3 = 0, \tag{35}$$

where $|\psi| = |\psi(r)|$, and $\mu_* = \mu - V_e$. The external confining potential V_e is assumed to be a negative constant, such that $\mu_* > 0$. Let us define $n_0 \equiv \mu_*/U_0$ which denotes the particle number density in uniform state at rest (when $U_0 > 0$). In fact, in a uniform state at rest (where $\nabla^2|\psi| = 0$ and $W_*(r) = 0$), we have $|\psi|^2 = \mu_*/U_0 = const$ from (35).

The length is normalized by the healing length ξ of (32), and the velocity by $\bar{v} = \hbar/m\xi$. Normalized amplitude A and radial coordinate ζ are defined by

$$A = |\psi|/\sqrt{n_0}, \qquad\qquad \zeta = r/\xi, \tag{36}$$

respectively. By these normalizations, the equation (35) reduces to

$$\nabla_\zeta^2 A - W_\delta^2(\zeta)\,A + A - A^3 = 0, \tag{37}$$

$$W_\delta(\zeta) \equiv \frac{1}{\zeta} - \frac{1}{\zeta}\,e^{-\eta^2} = \frac{1}{\zeta}\Big(1 - e^{-\eta^2}\Big), \qquad \eta = \zeta/\bar{\delta}, \quad \bar{\delta} = \delta/\xi. \tag{38}$$

In the present axisymmetric case, this is written as

$$A''(\zeta) + \frac{1}{\zeta}\,A'(\zeta) + A - W_\delta^2(\zeta)\,A - A^3 = 0. \tag{39}$$

The equation for the amplitude $A_p(\zeta)$ of a *potential vortex* (without rotational core) is immediately obtained from (39) by simply replacing $W_\delta(\zeta)$ with $1/\zeta$, which is given by

$$A_p''(\zeta) + \frac{1}{\zeta}\,A_p'(\zeta) + A_p - \frac{1}{\zeta^2}\,A_p - A_p^3 = 0, \tag{40}$$

(Ginzburg and Pitaevskii 1958, Lifshitz and Pitaevskii 1980, §30).

5.2. Power series solution for $\zeta \ll 1$

We seek a power series solution to Eq. (39) by using the series expansion of $W(\zeta)$:

$$W(\zeta) = \frac{1}{\zeta}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\eta^{2k}}{k!} = \beta\,\zeta - \frac{\beta^2}{2}\,\zeta^3 + \frac{\beta^3}{6}\,\zeta^5 + \cdots\cdots,$$

where $\beta = (\xi/\delta)^2$, and expressing A by the form: $A = b_0 + b_2\zeta^2 + b_4\zeta^4 + b_6\zeta^6 + \cdots$. Substituting these in (39) and replacing b_0 by A_0, the coefficients are determined by

$$b_2 = -\frac{1}{4}A_0\,(1 - A_0^2), \qquad b_4 = \frac{1}{16}A_0\left(\frac{1}{4} + \beta^2 - A_0^2 + \frac{3}{4}A_0^4\right),$$

$$b_6 = \frac{1}{36}(-b_4 + \beta^2 b_2 - \beta^3 A_0 + 3A_0 b_2^2 + 3A_0^2 b_4), \quad \cdots\cdots$$

If $A_0 = A(0) = 0$, we have the zero solution $A(\zeta) \equiv 0$. Hence in order to have a sensible solution, A_0 must be non-zero. This is remarkable, since it is well known that the amplitude $A_p(\zeta)$ of the potential vortex tends to zero like $b_1\,\zeta$ as $\zeta \to 0$ with b_1 a constant (see §5.3 (a)). This implies that the extended form of GP equation (30) permits a solution of a quantized vortex with a rotational core with non-zero number density $|\psi(0)|^2 \neq 0$.

5.3. Numerical solutions

(a) *Potential vortex* with quantized circulation: The amplitude $A_p(\zeta)$ is found numerically by solving the second-order differential equation (40) with the boundary conditions: $A_p(0) = 0$ and $A_p(\zeta) \to 1$ as $\zeta \to \infty$. In this solution, it is found that $A_p \approx b_1\zeta$ as $\zeta \to 0$ and $A_p \approx 1 - (1/2)\zeta^{-2}$ as $\zeta \to \infty$ (Lifshitz and Pitaevskii (1980) §30). The constant b_1 is determined numerically as 0.583190 to the first six digits.

(b) *Quantum vortex with rotational core*: By giving the value $A(0) = A_0$ at $\zeta = 0$ initially, the ordinary second-order differential equation (39) was solved. The solution was sought by varying the numerical value of $\bar{\delta} = \delta/\xi$ (the width of rotational core) with the *trial-and-error* method for a fixed A_0 so as to satisfy the condition at infinity: $A(\zeta) \to 1$ as $\zeta \to \infty$ (by the *Mathematica*). Corresponding to each particular value of A_0, the solution has been determined. Numerical parameters of some of the numerical solutions are listed in the Table 1.

Table 1. Parameters of numerical solutions, where $\Delta\overline{\mathcal{E}_\delta} \equiv \overline{\mathcal{E}_\delta} - \overline{\mathcal{E}_0}$ and $\overline{\mathcal{E}_\delta}$ is the total energy of a vortex of core size δ).

A_0	0	0.01	0.10	0.20	0.25	0.30	0.40
$\bar{\delta} = \delta/\xi$	0	0.009	0.086	0.169	0.209	0.249	0.330
$\Delta\overline{\mathcal{E}_\delta} \times 10^3$	0	−0.003	−0.213	−0.528	−0.539	−0.363	+0.710

Figure 1 shows the amplitude $A(\zeta)$ of the case $(A_0 = 0.25\,,\ \bar{\delta} = 0.20894)$ by a solid curve **a**, which is compared with the potential vortex plotted by a broken curve **b**. In this figure, the vorticity $\omega_c(\zeta)$ is also plotted by a dotted curve **c**, with a different vertical scale. It is seen that far from the core $(r > \xi)$, the two solutions **a** and **b** tends to be indistinguishable.

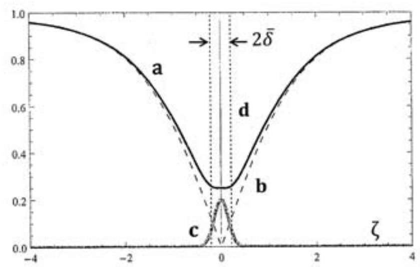

Figure 1. Amplitude $A(\zeta)$ of a quantum vortex with a rotational core ($A_0 = 0.25$, $\bar{\delta} = 0.20894$) is shown by the solid curve **a**, whereas the potential vortex $A_p(\zeta)$ by the broken curve **b**. The curve **c** at the bottom shows the vorticity $\omega_c(\zeta)$ with a different vertical scale. Two dotted vertical lines **d** show the size $\bar{\delta}$ of coaxial rotational core.

5.4. Energy of vortices with rotational core

Thus, we have found a family of solutions depending on $\bar{\delta}$ (or A_0) and having different values of total energy $\overline{\mathcal{E}_\delta}$. Using (24), we take its variation with keeping the total angular momentum $\overline{\boldsymbol{M}} = \int \tilde{\psi} \boldsymbol{M} \psi \, \mathrm{d}^3\boldsymbol{x}$ fixed with a fixed $\boldsymbol{\Omega}$. Hence we consider minimization of the first integral of $\mathcal{E}'_{GPr}$ only with ∇_0 replaced by $\mathcal{D}_w = \nabla + i(m/\hbar)\,\boldsymbol{w}_r$ of (28):

$$\mathcal{E}'_{GPr} = \int \mathrm{d}^3\boldsymbol{x} \Big[\frac{1}{2m} | - i\hbar \, (\nabla + i\frac{m}{\hbar}\boldsymbol{w}_r)\,\psi|^2 + \tfrac{1}{2}\, U_0\, |\psi|^4 \Big],$$

where the integration is taken over the plane perpendicular to the vortex axis, and the V_e term is neglected. Using the healing length ξ and the particle number density n_0 in the uniform state at rest, we normalize the wave function ψ and radial coordinate r like $\psi = \sqrt{n_0}\,\phi$ and $r = \xi\,\zeta$. Substituting these, we have

$$\mathcal{E}'_{GPr} = \frac{n_0\,\hbar^2}{2m} \int \Big(\xi^2\, |(\nabla + i\frac{m}{\hbar}\boldsymbol{w}_r)\,\phi|^2 + \frac{1}{2}\,|\phi|^4 \Big)\, \mathrm{d}^2\boldsymbol{\zeta}. \tag{41}$$

In non-uniform states in motion, the first term of the integrand is a dimensionless kinetic energy, denoted by $\mathcal{E}_k$. Writing as $\phi = A(r)\,\exp[i\,\varphi]$, the term $\xi\,[\nabla + i(m/\hbar)\boldsymbol{w}_r]\phi$ is

$$\xi\,[\nabla + i(m/\hbar)\boldsymbol{w}_r\,]\,\phi = (\nabla_\zeta A)\,\exp[i\,\varphi] + i\frac{m\xi}{\hbar}\Big(\frac{\hbar}{m}\nabla\varphi + \boldsymbol{w}_r\Big)\,A(r)\,\exp[i\,\varphi].$$

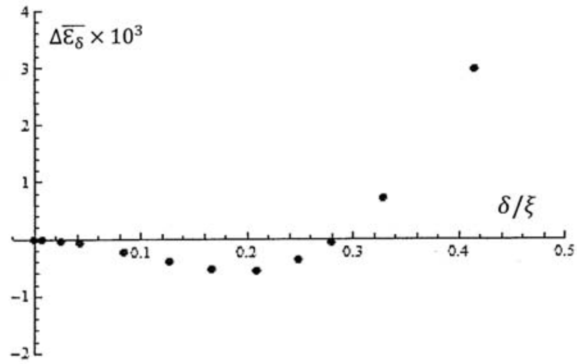

Figure 2. $\Delta\overline{\mathcal{E}_\delta} \times 10^3$ (vertical) plotted *vs.* the core size parameter δ/ξ (horizontal axis).

Defining $\Phi = (\hbar/m)\,\varphi$ and $\boldsymbol{v} = \nabla\Phi + \boldsymbol{w}_r$, the second term is expressed as $i(\boldsymbol{v}/\bar{v})\,A\,\exp[i\varphi]$ where $\bar{v} \equiv \hbar/m\xi$. Using $\boldsymbol{u} \equiv \boldsymbol{v}/\bar{v}$, we have

$$\mathcal{E}_k \equiv \xi^2\,|(\nabla + i\frac{m}{\hbar}\boldsymbol{w}_r)\phi|^2 = |\nabla_\zeta A|^2 + |\boldsymbol{u}|\,A^2 = |A'(\zeta)|^2 + u^2\,A^2, \qquad (42)$$

where $A'(\zeta) = \mathrm{d}A/\mathrm{d}\zeta$, and $u = W_\delta(\zeta)$ defined by (38). Denoting the integrand of (41) as $\mathcal{E}_\delta$ for a quantized vortex of rotational core of size δ, we have

$$\mathcal{E}_\delta = |A'(\zeta)|^2 + W_\delta^2(\zeta)\,A^2 + \frac{1}{2}\,A^4, \qquad W_\delta(\zeta) = (1/\zeta)(1 - e^{-\eta^2}). \qquad (43)$$

If we replace $W_\delta(\zeta)$ of the second term with $u = \zeta^{-1}$, we obtain the energy density $\mathcal{E}_0$ of the potential vortex of zero core-size:

$$\mathcal{E}_0 = |A_p'(\zeta)|^2 + \zeta^{-2}\,A_p^2 + \frac{1}{2}\,A_p^4.$$

where A_p denotes the amplitude of the potential vortex. Total angular momentum $\overline{M}$ is given by $(\overline{M_\delta}, 0, 0)$, and $\overline{M_\delta}$ by $n_0\hbar\int \mathcal{M}_\delta\, 2\pi\zeta\mathrm{d}\zeta$, where

$$\mathcal{M}_\delta = \zeta\,u\,A^2, \qquad \mathcal{M}_0 = A_p^2.$$

Using the numerical solutions obtained in the previous section §5.3, we can estimate the total energy $\overline{\mathcal{E}_\delta} = \int \mathcal{E}_\delta\; 2\pi\zeta\mathrm{d}\zeta$. In this numerical estimate, the upper end of integration ζ_m is chosen so as to fix the integration value $\int_0^{\zeta_m} \mathcal{M}_\delta\,\zeta\mathrm{d}\zeta$ of total angular momentum. The ζ_m-value depends on δ and was determined so as to take the same total angular momentum as that of the potential vortex of $\delta = 0$ for which $\zeta_m = 5$ was chosen.

Figure 2 shows the variation $\Delta\overline{\mathcal{E}_\delta} = \overline{\mathcal{E}_\delta} - \overline{\mathcal{E}_0}$ versus the normalized size δ/ξ of rotational core. It is interesting to find that there exists a minimum of $\Delta\overline{\mathcal{E}_\delta}$ at $\delta/\xi \approx 0.2$, which is lower than $\Delta\overline{\mathcal{E}_\delta} = 0$ of the potential vortex. This property is kept unchanged qualitatively even if a larger value of $\zeta_m = 10$ was taken. Thus, from the energy consideration, it is probable that a vortex with a rotational core of $\delta/\xi \approx 0.2$ is generated in reality. Note that, for $\zeta > \xi$ (*i.e.* out of inner core), difference of the two solutions of $\delta \approx 0.2\xi$ and the potential vortex of $\delta = 0$ is very small to such a degree that challenges experimental resolution. Relative variation of $|\Delta\overline{\mathcal{E}_\delta}|/\overline{\mathcal{E}_\delta}$ is less than 0.001 when $\Delta\overline{\mathcal{E}_\delta} < 0$, where $\overline{\mathcal{E}_\delta} \approx 5.816$.

6. Summary and discussions

It is proposed that the Gross-Pitaevskii equation governing a non-ideal (interacting) Bose gas admits *rotational* flows. This is verified by applying a quantum-mechanical scenario to the equation without incurring essential change in the equation, or alternatively by introducing a local gauge transformation to the GP equation, with replacing the standard momentum operator with a covariant derivative including a gauge field.

By this reformulation, a superfluid placed in a rotating vessel is able to have a solid body rotation and also to have a meniscus approximated by a parabolic profile. The solid body rotation is accompanied by density increase proportional to the square of its angular velocity. This density increase is supported by the experimental observation of Andronikashvili and Tsakadze (1965a, b). The parabolic meniscus was also observed by various experiments cited in the article (*b*) of §1.1.

In addition, the reformulation allows a vortex of quantized circulation to have a *coaxial rotational core.* It is shown by numerical analysis that a vortex of core size $\delta \approx 0.2\xi$ has the lowest total energy which is lower than that of a potential vortex without rotational core.

If there is a rotational core in the vortex, the density does not vanish at its center. Most observations report that there exist certainly dips in the density profiles. However, to the author's knowledge, no experiment reports that the density vanishes exactly at the vortex center. If there is a rotational core, there may be observable consequences on the dispersion relation of Kelvin waves (helical perturbations of the vortex), or on vortex reconnection in superfluid.* This is considered in some detail next.

The vortex with a rotational core obtained in the present study has a core of size $\delta \approx 0.2\xi$. The Gaussian profile (34) of the core vorticity is approximated by a constant vorticity circular core of radius δ of the same strength in the following order estimate. Suppose that the vortex filament is deformed into a helix of a wave number k. Then, according to Thomson (1880), the dispersion relation is given by

$$(\omega_\delta - \omega_0)/\omega_0 = 1/(4L), \qquad \omega_0 = -\tfrac{1}{2}\,\gamma\, k^2\, L, \qquad L = \ln(1/k\delta) + \alpha$$

for very small $k\delta$ and $\alpha = 0.1159$, where ω_0 and ω_δ are the angular frequency of a vortex of hollow core and that of a rotational core respectively (both being roughly assumed to be the same size δ for simplicity reason). The negative sign of the ω_0-expression means rotation in the reverse direction to that of fluid swirling around the filament.

According to the data of recent observation of Fonda *et al* (2012) in superfluid helium, we estimate $L \approx 14$. Hence the relative difference is estimated as $(\omega_\delta - \omega_0)/\omega_0 \approx 0.02$. This difference is challengingly small, *i.e.* it may be too small to be detected by currently available experimental data. Another observable phenomenon may be the vortex reconnection. This is important, but is out of our scope, because the vortex reconnection is regarded as caused by diffusion effect, or by finite-temperature effect.

* This is one of the comments from a reviewer.

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